INTRODUCTION TO TOPOLOGICAL SOLITONS

Plan:
1) KINKS & DOMAIN WALLS
2) TOPOLOGY, GENERAL INTRODUCTION
3) SIGMA MODELS
4) VORTICES
5) MONOPOLES
6) INSTANTONS
7) DYONS

Books: "TOPOLOGICAL SOLITONS" Manton & Sutcliffe (2004)

"ASPECTS OF SYMMETRY" Coleman (1985)

"SOLITONS AND INSTANTONS" Rajaraman (1982)

Selection of Papers/Lectures:

MONOPOLES, INSTANTONS & CONFINEMENT
't Hooft, Bruckmann, hep-th/0010225

TASI LECTURES ON SOLITONS
D. Tong, hep-th/0509216

ABC OF INSTANTONS
Vainshtein, Zakharov, Novikov, Shifman

1 KINKS & DOMAIN WALLS

1+1 DIMENSIONS $\quad g_{\mu\nu} = \begin{pmatrix} +1 & \\ & -1 \end{pmatrix}$

ONE REAL SCALAR FIELD $\phi(x,t)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi)$$

$$L = T - V \qquad T = \int_{-\infty}^{+\infty} dx\ \frac{1}{2}\dot{\phi}^2$$

$$V = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2}\phi'^2 + U(\phi)\right)$$

E-L EQUATIONS

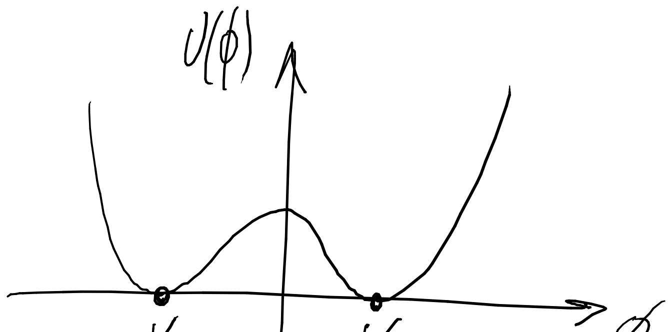
$$\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} - \frac{\delta\mathcal{L}}{\delta\phi} = 0 \Rightarrow \boxed{\Box\phi + \frac{\delta U}{\delta\phi} = 0}$$

Static Configurations

$$\mathrm{Vac} = \{\phi,\ \phi' = \dot{\phi} = 0,\ U(\phi) = 0\}$$

EXAMPLE

$$U(\phi) = \lambda(v^2 - \phi^2)^2$$

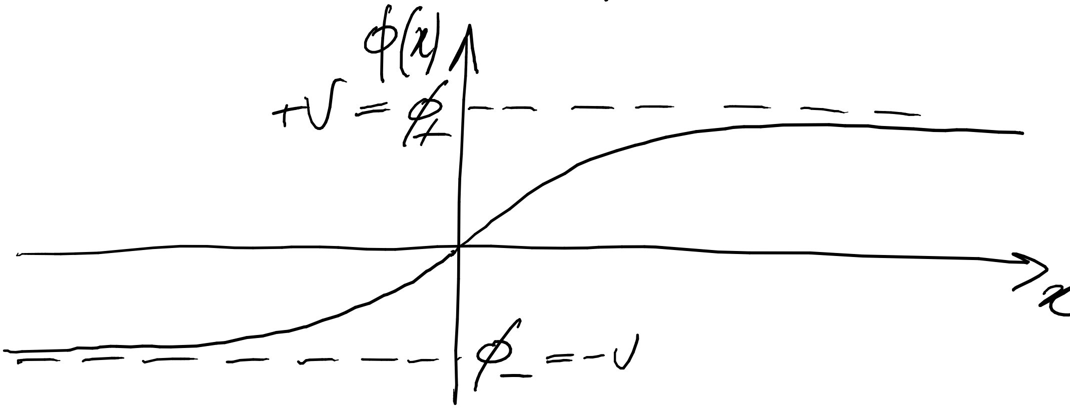


$\mathbb{Z}_2$ SYMMETRY

$$\phi \to -\phi$$

$\phi_+(x,t) = v$ OR $\phi_-(x,t) = -v$ ARE SOLUTIONS TO E_L EQUATIONS.

THERE IS ALSO A STATIC SOLUTION (BUT x DEPENDENT!) THAT INTERPOLATES BETWEEN ϕ_+ AND ϕ_-



THIS IS CALLED A KINK!

FIRST LET'S SEE THAT IT REALLY EXISTS

E_L

$$\Box\phi + \frac{\delta U}{\delta \phi} = 0 \underset{\text{STATIC}}{\Longrightarrow} \phi'' - \frac{\delta U}{\delta \phi} = 0$$

$$\phi'' - 4\lambda\phi(v^2 - \phi^2) = 0$$

THIS ODE MUST BE SOLVED WITH BOUNDARY CONDITIONS

$$\phi(-\infty) = -v \,, \quad \phi(+\infty) = +v$$

WE EXPECT SOLUTION TO EXIST...

ENERGY MINIMIZATION:

$$E = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2}\phi'^2 + U(\phi) \right)$$

NOW WE USE THE SO CALLED "BOGOMOLNY TRICK"

$$\left(\frac{1}{\sqrt{2}}\phi' \pm U(\phi)^{1/2} \right)^2 \geq 0$$

THIS IS ALWAYS TRUE!

$$\int_{-\infty}^{+\infty} \left(\quad \right)^2 dx \geq 0$$

OF COURSE!

$$\underbrace{\int_{-\infty}^{+\infty} dx \left(\frac{1}{2}\phi'^2 + U(\phi) \right)}_{= E} \pm \underbrace{\int_{-\infty}^{+\infty} \sqrt{2}\,\phi' U(\phi)^{1/2} dx}_{} \geq 0$$

THIS DOES NOT DEPEND ON $\phi(x)$ BUT JUST ON THE BOUNDARY VALUES!

$$\phi' dx = d\phi$$

$$E \geq \mp \int_{\phi_-}^{\phi_+} d\phi \sqrt{2U(\phi)}$$

"BOGOMOLNY BOUND"

SIGN MUST BE CHOOSEN TO HAVE MORE RESTRICTIVE BOUND.

INTRODUCE A "SUPERPOTENTIAL"

$$U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$$

$$\int_{\phi_-}^{\phi_+} d\phi \sqrt{2U} = W(\phi_+) - W(\phi_-)$$

SO:
$$E \geq \left| W(\phi_+) - W(\phi_-) \right|$$

THE BOUND IS SATURATED IF

$$\phi' \pm \sqrt{2U(\phi)} = 0 \qquad \text{(Bogomolny's Equation)}$$

IF $\phi_+ > \phi_- \Rightarrow$ CHOOSE $\ominus$ SIGN

BOGO EQUATION IS 1^{st} ORDER

BOGO $\Rightarrow$ E-L

$$\left(\phi' \pm \sqrt{2U}\right)' = 0 \Rightarrow \phi'' \pm \frac{1}{\sqrt{2}} \frac{\frac{dU}{d\phi}\phi'}{\sqrt{U}} = 0$$

USE AGAIN BOGO.

$$\phi'' - \frac{dU}{d\phi} = 0$$

THE CONVERSE IS NOT TRUE IN GENERAL

E-L $\not\Rightarrow$ BOGO   NOT ALL SOLUTIONS TO E-L SATURATES THE BOUND.

$U(\phi) = \lambda \left(v^2 - \phi^2\right)^2$ QUARTIC POTENTIAL

$\mathbb{Z}_2$ SYMMETRY

$\phi_- = -v$

$\phi_+ = v$

$E \geq \underbrace{\left|W(\phi_+) - W(\phi_-)\right|}$

$W(\phi) = \sqrt{2\lambda}\left(v^2\phi - \frac{1}{3}\phi^3\right)$

$U = \frac{1}{2} W'^2$

$$E_{BOUND} = 2 \cdot \sqrt{2\lambda}\left(\frac{2}{3}v^3\right)$$

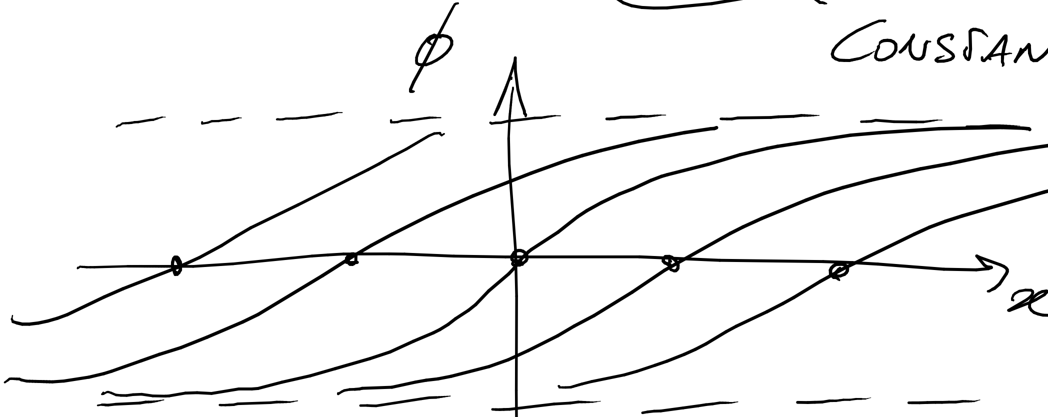
$$= \frac{4\sqrt{2}}{3}\sqrt{\lambda}\, v^3$$

BOGO EQUATION

$$\phi' = \sqrt{2\lambda}\left(v^2 - \phi^2\right) \qquad \int \frac{d\phi}{\sqrt{2\lambda}\left(v^2 - \phi^2\right)} = \int dx$$

$$\phi = v \tanh\left(\sqrt{2\lambda}\, v (x - a)\right)$$

a IS INTEGRATION CONSTANT

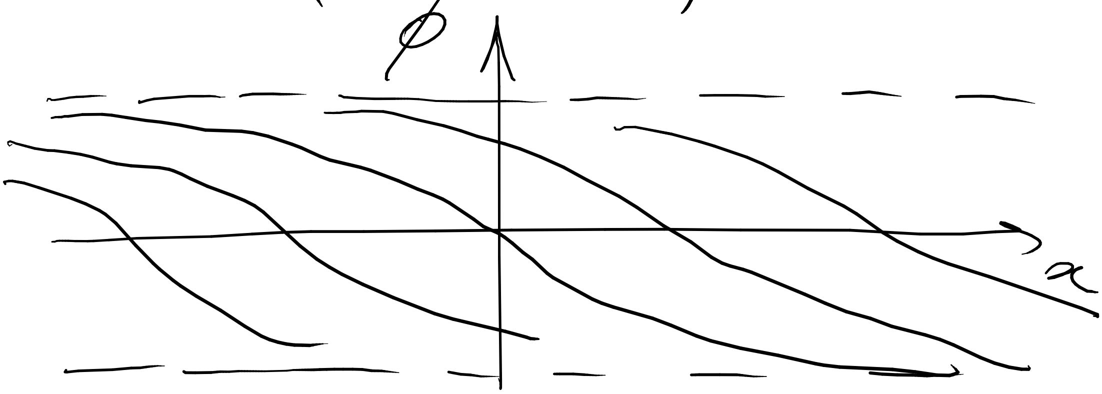


CHOOSING INSTEAD $\phi_- = v$, $\phi_+ = -v$
I HAVE TO SOLVE

$$\phi' = -\sqrt{2\lambda}\,(v^2 - \phi^2)$$

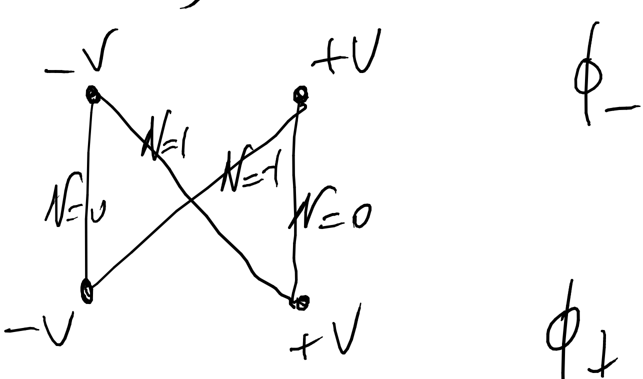
(RELATED TO THE PREVIOUS BY $\phi \to -\phi$)

$$\phi = v \tanh\left(\sqrt{2\lambda}\, m\, (a - x)\right)$$



THE BOGO BOUND IS THE SAME.

$$N = \frac{1}{2v}\int_{-\infty}^{+\infty} \phi'\, dx$$ IS A CONSERVED QUANTITY



FINITE ENERGY CONFIGURATIONS SPLIT INTO DISTINCT TOPOLOGICAL SECTORS.

INTERPRETATION

$[\phi] = m^0, \ [v] = m^0, \ [\lambda] = m^2$

$E = \frac{4\sqrt{2}}{3}\sqrt{\lambda}\, v^3$ IS A MASS

- ARE TWO LOCALIZE LUMPS OF ENERGY WITH TOTAL MASS E_{BOUND}.
- KINK AND ANTI-KINK

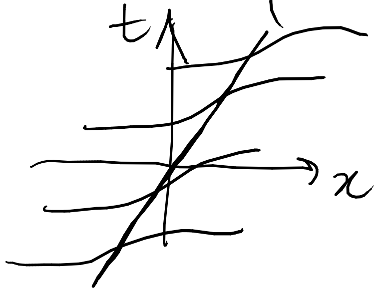
THEY HAVE TWO OPPOSITE CHARGE RESPECT TO THE TOPOLOGICALLY CONSERVED QUANTITY N.

- THEY CAN BE PLACED AT ANY POINT

$\phi \propto \tanh(x-a) \qquad \phi \propto \tanh(a-x)$

- THEY CAN BE BOOSTED AT FINITE VELOCITY

$$\phi(x,t) = v \tanh\left(\sqrt{\lambda}\, v\, \gamma(x - vt - a)\right)$$



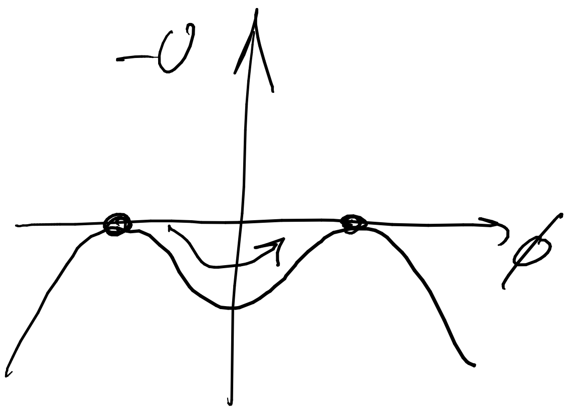
THEY REALLY LOOK LIKE PARTICLES!

E-L EQUATION FOR STATIC SOLUTIONS

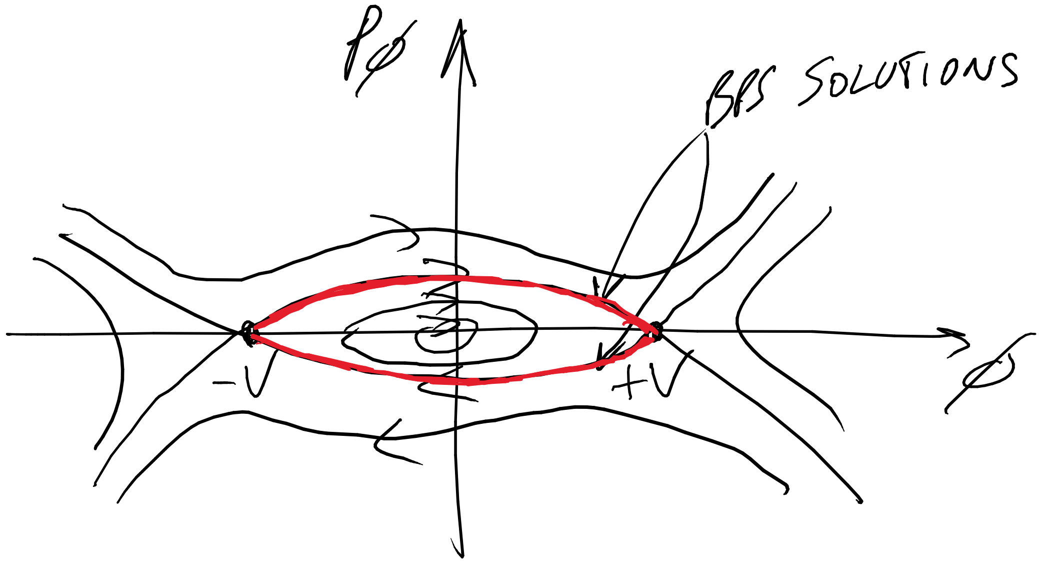
$$\phi'' - \frac{dU}{d\phi} = 0$$

IS THE SAME EQUATION

OF A PARTICLE MOVING IN TIME ON AN INVERTED POTENTIAL



KINK AND ANTI-KINK INTERPOLATE AT $t \to \pm\infty$ BETWEEN THE TWO MAXIMA



PHASE SPACE OF MECHANICAL SYSTEM

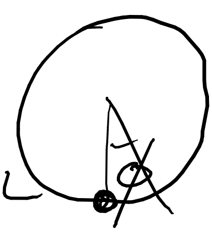
$\Rightarrow$ NO STATIC SOLUTIONS OF MULTIPLE KINKS.

SINE-GORDON KINKS

$U(\phi) = 1 - \cos\phi$
$\quad = 2\sin^2\frac{\phi}{2}$

VACUA ARE LABELED BY AN INTEGER n, $\phi = 2\pi n$

INVARIAN UNDER $\phi \to \phi + 2\pi$ IN FACT WE CAN THINK OF TARGET SPACE AS A CIRCLE



S^1

OR A GRAVITATIONAL POTENTIAL

SINE-GORDON MODEL $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - (1-\cos\phi)$

$$N = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi'\, dx$$

NOW I HAVE A LOT OF DIFFENT TOPOLOGICAL SECTORS

$$E \geq E_{BOUND} = \int_0^{2\pi n} 2\left|\sin\frac{\phi}{2}\right| d\phi = 8|n|$$

BOGO EQUATION

$$\phi' = \pm 2 \sin \phi/2$$

$$\int \frac{d\phi}{2 \sin \phi/2} = \int dx$$

$$y = \tan \phi/4$$

$$dy = \frac{d\phi}{4} \frac{1}{\cos^2 \phi/4}$$

$$\int \frac{dy \cdot 2 \cdot \cos^2 \phi/4}{\sin \phi/2}$$

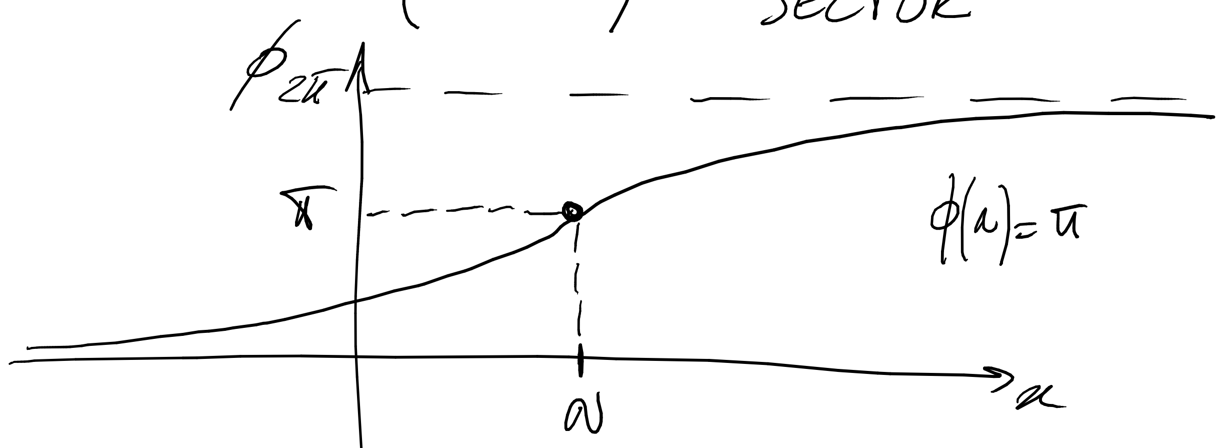
$$\sin \phi/2 = 2 \sin \phi/4 \cos \phi/4$$

$$\int \frac{dy}{y} = \int dx$$

$$\log(\tan \phi/4) = x - a$$

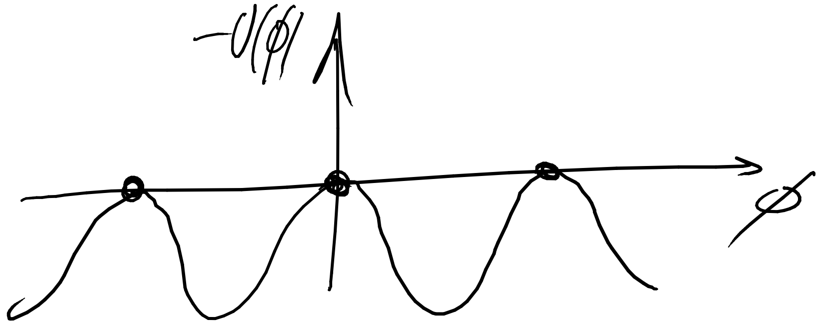
$$\phi = 4 \operatorname{Arctan}\left(e^{x-a}\right)$$

ONE SOLITON SECTOR

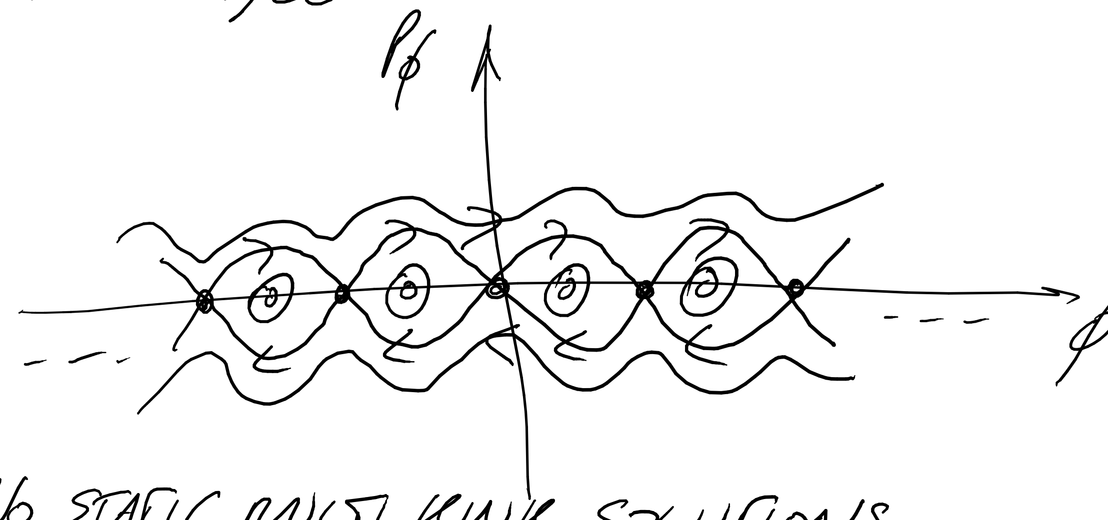


THIS IS THE KINK INTERPOLATING BETWEEN ANY $\phi_- = n$, $\phi_+ = n+1$.
THE ANTI-KINK INTERPOLATES BETWEEN $\phi_- = n+1$, $\phi_+ = n$,

IN THE ANALOG MECHANICAL SYSTEM THE INVERTED POTENTIAL IS



THE TRAJECTORIES ON PHASE SPACE

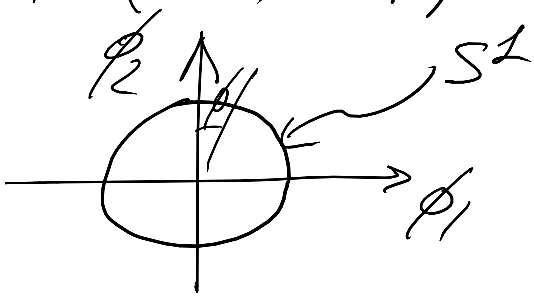


NO STATIC MULTI KINK SOLUTIONS UNLESS THEY ARE INFINITE.

SG IS INTEGRABLE AS A 1+1 MODEL

SINE-GORDON REVISITED

$$\vec{\phi} = (\phi_1, \phi_2) = (\sin\phi, \cos\phi)$$



$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - (1-\cos\phi)$$

CAN BE REWRITTEN IN (ϕ_1, ϕ_2) AS FOLLOWS

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} + \lambda(1-\vec{\phi}\cdot\vec{\phi}) - (1-\phi_2)$$

λ IS A LAGRANGE MULTIPLIER

$$\frac{\delta\mathcal{L}}{\delta\lambda} = (1-\vec{\phi}\vec{\phi}) = 0$$

THIS FORCES $\vec{\phi}$ TO LIVE ON THE S^1 CIRCLE

$1-\cos\phi$

FINITE ENERGY SOLUTIONS

$$\phi(x)$$

$$\mathbb{R} \cup \{\infty\} \longrightarrow S^1$$

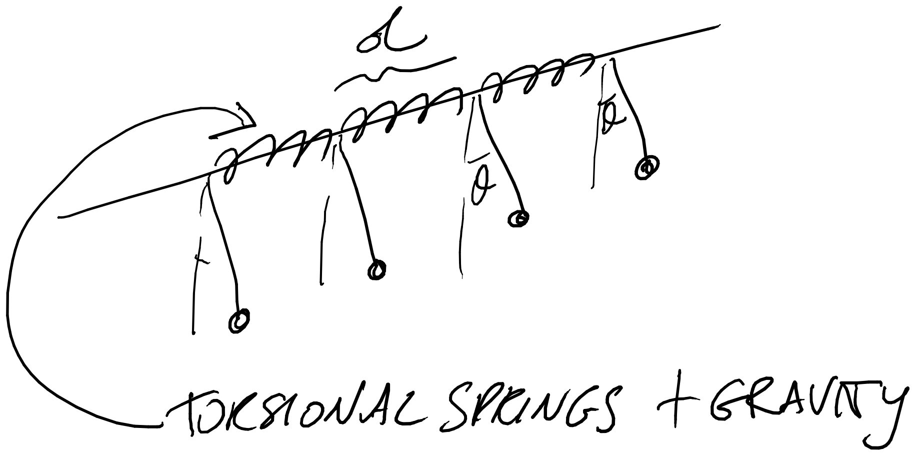
$$\phi(+\infty) = \phi(-\infty) = 0 \ (\mathrm{mod}\ 2\pi n)$$

SOLUTIONS (FINITE ENERGY!) ARE CLASSIFYED BY THE ELEMENTS OF THE HOMOTOPY GROUP

$$\pi_1(S^1) = \mathbb{Z}$$

$$N = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, \epsilon_{ab} \phi_a' \phi_b$$

PHYSICAL REALIZATION OF S-G MODEL



$$L = \sum_n \left(\frac{\overset{ml}{I}}{2} \dot{\theta}_n^2 - \frac{c}{2}\left(\theta_n - \theta_{n-1}\right)^2 - mgl\left(1-\cos\theta_n\right)\right)$$

CONTINUUM LIMIT (CONSIDER ONLY FLUCTUATIONS AT SCALE >> d)

$$L = \frac{ml}{d}\int dx \left(\frac{\dot{\phi}^2}{2} - \frac{\phi'^2}{2}\frac{cd^2}{ml} - g\left(1-\cos\phi\right)\right)$$

$$S = \int dt\, L$$

SPEED OF "LIGHT"

$$v = \sqrt{\frac{cd^2}{ml}}$$

$$\phi = 4\,\mathrm{Arctan}\left(\frac{v \sinh(\gamma x)}{\cosh(\gamma v t)}\right) \qquad (KK)$$

$$\phi = 4\,\mathrm{Arctan}\left(\frac{\sinh(\gamma v t)}{v \cosh(\gamma x)}\right) \qquad (K\bar{K})$$

INTERPRET THESE SOLUTIONS

QUANTUM MECHANICS & SOLITONS

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi) \qquad U(\phi) = \lambda(v^2-\phi^2)^2$$

1+1

$[\phi] = m^0$

$[v] = m^0$

$[\lambda] = m^2$

$[E_{BOUND}] = m$

PARTICLE MASS

3+1

$[\phi] = m$

$[v] = m$

$[\lambda] = m^0$

$[E_{BOUND}] = m^3$

DOMAIN WALL TENSION

$$\phi = v + \delta\phi$$

$$U(\phi) \simeq 4\lambda v^2 \delta\phi^2 + \dots \qquad \delta\phi \ll v$$

MASS PERTURBATIVE PARTICLE

$$m_\phi = 2\sqrt{2\lambda}\, v$$

$v \gg 1$

CLASSICAL LIMIT.

$e^{iS/\hbar}$

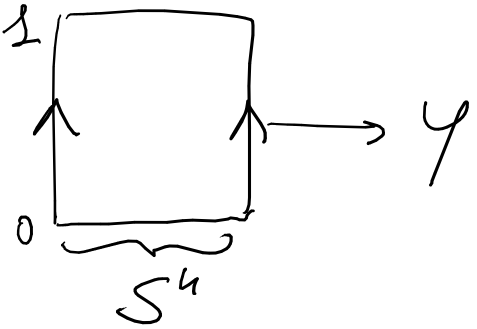
$\hbar = 1$

② TOPOLOGY, GENERAL INTRODUCTION

$\varphi : S^n \longrightarrow Y$ $\qquad$ " $\varphi(x_0) = y_0$

S^n: n-DIMENSIONAL SPHERE $\qquad$ BASED MAP$_n$

Y: MANIFOLD (WITH NO BOUNDARIES)



$I = [0,1]$

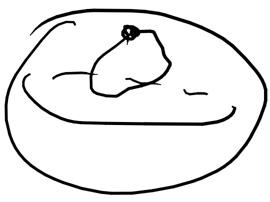
$\tilde{\varphi} : S^n \times [0,1] \longrightarrow Y$

$\tilde{\varphi}\big|_{I=0} = \varphi_0$

$\tilde{\varphi}\big|_{I=1} = \varphi_1$

φ_0 AND φ_1 ARE HOMOTOPICALLY EQUIVALENT IF $\tilde{\varphi}$ CONTINUOUS EXISTS

$\varphi : S^1 \longrightarrow S^2$



EVERY MAP IS HOMOTOPIC TO IDENTITY MAP

HOMOTOPY IS EQUIVALENCE RELATION

$\Pi_n(Y)$ IS ALSO A GROUP FOR $n \geq 1$

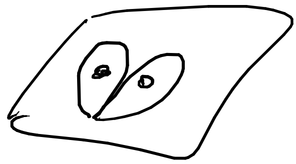
$\pi_0(Y)$ # DISCONNECTED COMPONENTS OF Y

$\pi_1(Y)$ FUNDAMENTAL GROUP

MAY BE NON ABELIAN

$\pi_1(S^1) = \mathbb{Z}$

$\pi_1(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z}$



$\pi_1(\mathbb{R}/\{0\},\{1\})$

NON ABELIAN

$\pi_{n>1}(Y)$ ALWAYS ABELIAN

$\theta \longrightarrow \mathbb{R}$ $\qquad x \simeq x + 2\pi p$

$S^1 \longrightarrow S^1$

$f(2\pi) = 2\pi k + f(0)$ $\qquad k$: WINDING NUMBER

IF f_0, f_1 HAVE THE SAME k

$\tilde{f} = (1-\tau) f_0 + \tau f_1$ INTEROPOLETE CONTINUOUSLY BETWEEN THE TWO.

AS A GROUP IS ADDITIVE IN K.

ALGEBRAIC TOPOLOGY STUDIES THE HOMOTOPY GROUPS OF VARIOUS SPACES

$$\pi_n(S^n) = \mathbb{Z} \quad \forall\, n \geq 1$$

$$\left\{ \begin{array}{l} \pi_1(S^1) \\ \pi_2(S^2) \\ \pi_3(S^3) \end{array} \right.$$

$$\pi_n(S^d) = 1 \quad \text{IF } n < d$$

$$\pi_3(S^2) = \mathbb{Z}$$ HOPF CHARGE FIRST EXAMPLE OF NON TRIVIAL HIGHER TOPOLOGY

$$\pi_{n+1}(S^n) = \mathbb{Z}_2 \quad n \geq 3 \qquad \left(\pi_4(S^3) = \mathbb{Z}^2\right)$$

SPIN SKYRMION

COSET SPACES

$$H \subset G \qquad G \text{ LIE GROUP}$$

G/H COSET (LEFT OR RIGHT)

$$\pi_2(G/H) = \pi_1(H) \qquad \pi_1(G/H) = \pi_0(H)$$

$$\pi_2\left(SU(2)/U(1)\right) = \pi_2(S^2) = \mathbb{Z}$$

$$\shortparallel$$

$$\pi_1(U(1)) = \mathbb{Z}$$

$\varphi: S^4 \to S^4 \qquad \pi_4(S^4) = \mathbb{Z}$

$[k] \in \pi_4(S^4) \implies \deg \varphi_{[k]} = k$

WHEN $\varphi: X \to Y$ WITH $\dim X = \dim Y = d$

WE CAN DEFINE THE DEGREE OF A MAP

$\Omega = \beta(y)\, d^d y$ VOLUME FORM ON Y

$\int_Y \Omega = 1$ NORMALIZATION

$X \xrightarrow{\varphi} Y$

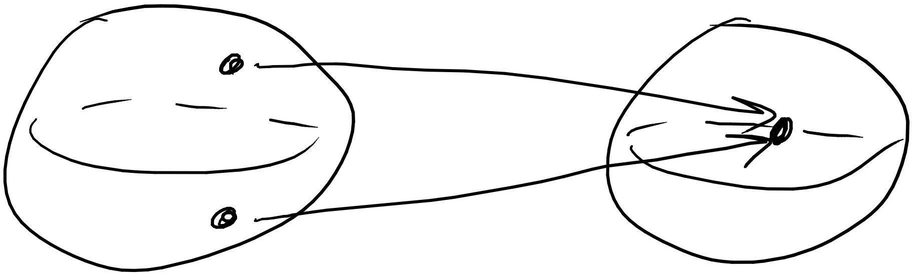
$\varphi^*(\Omega) \leftarrow \Omega$ PULL-BACK OF THE FORM

$\varphi^*(\Omega) = \beta(y(x))\, J(x)\, d^d x$ JACOBIAN

$= \det\left(\frac{\partial \varphi}{\partial x}\right)$

$\deg \varphi = \int_X \varphi^*(\Omega)$

* # TIMES Y IS COVERED BY THE MAP



\# PRE-IMAGES OF "ALMOST ANY" POINTS

$\deg \psi$ = INTEGER = HOMOTOPY INVARIANT

$$\boxed{k = \deg \psi}$$

$$\psi : S^1 \longrightarrow S^1$$

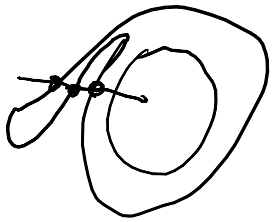
$\pi_1(S^1) = \mathbb{Z}$

SINE-GORDON MODEL

$$\Omega = \frac{1}{2\pi} d\phi$$

$$\int_\psi \Omega = \frac{1}{2\pi} \int_0^{2\pi} d\phi = 1$$

$$\deg \phi = \int_X \psi^*(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, \phi' = k \in \mathbb{Z}$$

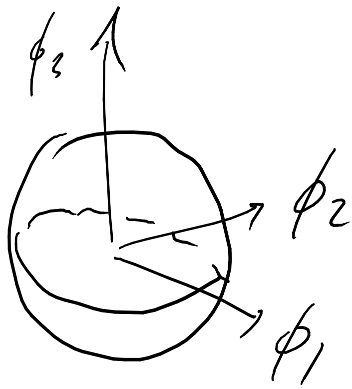


COUNTING PRE-IMAGES WITH A SIGN!

GENERALIZATION TO HIGHER DIMENSION

$$\phi : \mathbb{R}^2 \cup \{\infty\} \longrightarrow S^2$$

$$\vec{\phi}\cdot\vec{\phi} = 1 \qquad \phi = (\phi_1, \phi_2, \phi_3)$$



SIGMA-MODEL
TARGET SPACE S^2

$\pi_2(S^2)$ CLASSIFY MAPS THAT HAVE FIXED VALUE AT INFINITY (ENERGETIC ARGUMENT)

$$\underbrace{\deg \phi = \frac{1}{8\pi}\int \epsilon_{ij}\,\vec{\phi}\cdot\left(\partial_i\vec{\phi}\wedge\partial_j\vec{\phi}\right) d^2x}_{\text{TOPOLOGICAL INVARIANT}}$$

$$N = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \epsilon_{ab}\,\phi_a'\,\phi_b\,dx$$

$$N = \frac{1}{8\pi}\int d^2x\, \epsilon_{abc}\,\epsilon_{ij}\,\phi^a\,\partial_i\phi^b\,\partial_j\phi^c$$

...

$$\underbrace{\frac{1}{4\pi}}_{\text{VOL } S^2} \quad \underbrace{\frac{1}{2}\epsilon_{ij}\epsilon_{abc}\phi^a\partial_i\phi^b\partial_j\phi^c}_{\text{JACOBIAN}}$$

FIND THE GENERALIZATION

$$\mathbb{R}^3 \cup \{\infty\} \longrightarrow S^3$$

DERRICK'S THEOREM

GENERA THEOREM / NON EXISTENCE THEOREM IN FLAT SPACE

SCALING $x \to \lambda x$

$$E[\varphi(x)] = \dots \quad \left(\text{Ex. } \int_{-\infty}^{+\infty} dx \left(\frac{\phi'^2}{2} + U(\phi)\right)\right)$$

$$\varphi(x) \to \varphi(\lambda x)$$

IF $\varphi(x)$ IS SOLUTION TO E-L, THEN IT MUST ALSO BE STATIONARY UNDER λ SCALING. $\mathbb{R}^{d,1}$

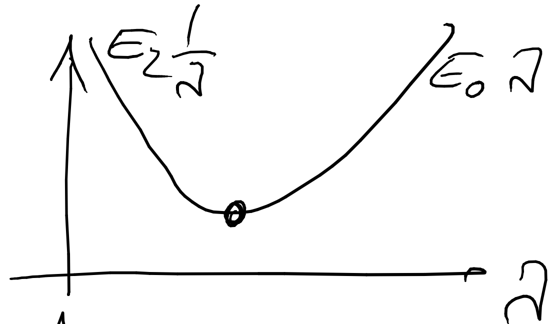
$$E = \int d^dx \left(\partial_i\phi\partial_i\phi + U(\phi)\right) = E_2 + E_0$$

\# DERIVATIVES

$$E(\lambda) = \lambda^{d-2} E_2 + \lambda^d E_0$$

d^dx $\qquad$ $\partial_i \partial^i$

$d=1$



$d=2$

$E_0 = 0$ SIGMA-MODEL LUMPS

$d=2$ $\qquad$ $E_0 + E_2 + E_4$ $\qquad$ BABY-SKYRME MODEL

4 DERIVATIVE TERM

$d=3$ $\qquad$ $(E_0) + \underbrace{E_2 + E_4}_{\text{Skyrme theory}}$

ALTERNATIVE TO HIGHER DERIVATIVE TERMS WE CAN USE ALSO GAUGE FIELDS

$$E=\int d^dx\left(\underset{\substack{\downarrow\\ E_4}}{F^2}+\underset{\substack{\downarrow\\ E_2}}{|D\phi|^2}+\underset{\substack{\downarrow\\ E_0}}{U(\phi)}\right)$$

$D=\partial+iA$ $\quad$ A IS LIKE A DERIVATIVE UNDER SCALING

$$E(\lambda)=\lambda^{d-4}E_4+\lambda^{d-2}E_2+\lambda^{d}E_0$$

$d=2$ $\quad$ VORTICES (LIKE BABY-SKYRME MODEL)

$d=3$ $\quad$ $\frac{1}{\lambda}E_4+\lambda E_2$ $\quad$ MONOPOLE

$d=4$ $\quad$ E_4 $\quad$ INSTANTON (NO SCALE)

MODULI SPACE APPROXIMATION

IN "CERTAIN CASES" WE A CONTINUOS FAMILY OF SOLUTIONS TO STATIC EQUATIONS MODULI SPACE IS USUALLY CALLED.

$\phi(x; a)$ a PARAMETERIZE THE MODULI SPACE

$$S = \int dt\, d^d x\, \mathcal{L}(\phi, \partial\phi)$$

$\phi = \phi(x; a(t))$ MODULI SPACE APPROX.

AT EVERY t THE SNAPSHOT IS A STATIC SOLUTION.

INSERTING IN THE ACTION WE GET

$$S = \int dt \left(\frac{1}{2} g_{ij}\, \dot{a}_i\, \dot{a}_j + \ldots \right)$$

METRIC IN THE MODULI SPACE

IS A VALID APPROXIMATION WHEN THE MOTION IS SUFFICIENTLY SLOW.

$V(a) = 0$ BECAUSE THERE ARE NO STATIC FORCES. FORCES ARE ONLY VELOCITY DEPENDENT.

$$\partial_t \left(\frac{\delta L}{\delta \dot{a}} \right) - \frac{\delta L}{\delta a} = 0$$

$$\partial_t (g_{ij} \dot{a}_j) - \frac{1}{2} \dot{a}_i \dot{a}_j \frac{\delta g_{ij}}{\delta a_k} = 0$$

$$g_{ij} \ddot{a}_j + \partial_k g_{ij} \dot{a}_j \dot{a}_k - \frac{1}{2} \partial_k g_{ij} \dot{a}_i \dot{a}_j = 0$$

$$g_{ij} \left(\ddot{a}_j + \underbrace{\Gamma^j_{kl}}_{\frac{1}{2} g^{ij} (\partial_k g_{li} + \partial_l g_{ki} - \partial_i g_{kl})} \dot{a}_k \dot{a}_l \right) = 0$$

LEVI-CIVITA CONNECTION

Moduli space approx.

$\Leftrightarrow$ GEODESIC IN MODULI SPACE

SPONTANEOUSLY BROKEN SYMMETRIES

$\Downarrow$

MODULI SPACE

$\phi(x-a)$ IS A SOLUTION FOR A KINK IN 1+1 DIMENSIONS, $\forall a$

a IS A MODULI SPACE, IT IS A CONSEQUENCE OF S.B. TRANSLATION INVARIANCE.

$\phi(\gamma(x-vt))$ IS EXACT SOLUTION

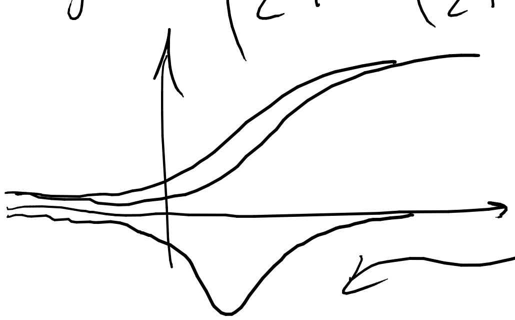
LORENTZ BOOST

FOR SLOW MOTION $v \ll 1$

$\simeq \phi(x-vt)$ $\rightarrow$ $a=vt$ MOTION IN THE MODULI SPACE

NEGLECT LORENTZ CONTRACTION.

$$S=\int dx\,dt\left(\frac{1}{2}\dot{\phi}^2-\left(\frac{1}{2}\phi'^2+U(\phi)\right)\right)$$



$\frac{\delta\phi}{\delta a}$ ZERO MODE

$\phi(x, a(t))$

BOGO. BOUND
SATURATED

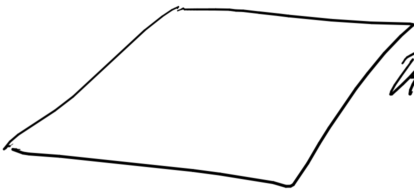
$$\partial_t \phi = \frac{\delta \phi}{\delta a} \dot{a}$$

$$S = \int dt \left(-M + \frac{1}{2} \dot{a}^2 \underbrace{\int dx \left(\frac{\delta \phi}{\delta a} \right)^2}_{\substack{\parallel \\ M}} \right)$$

METRIC ON THE
MODULI SPACE

3) SIGMA MODEL LUMPS

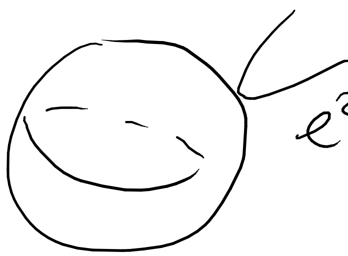
O(3) SIGMA MODEL



$\mathbb{R}^{d,1} \longrightarrow S^2 \qquad \vec{\phi} = (\phi_1, \phi_2, \phi_3)$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \lambda (1 - \vec{\phi} \cdot \vec{\phi})$$

$$\mathcal{L} = \frac{1}{2e^2} \lambda^2 + \lambda (1 - \vec{\phi} \cdot \vec{\phi}) \qquad \lambda = e^2 (1 - \vec{\phi} \cdot \vec{\phi})$$



$e^2 \to \infty$ MASS GOES TO INFINITY

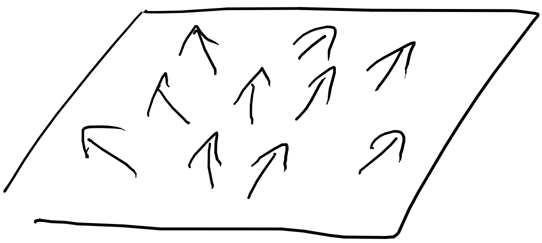
$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \quad \text{with } \vec{\phi} \cdot \vec{\phi} = 1$$

ALL THE INTERACTION IS CAUSED BY THE CONSTRAINT

$\phi \to M\phi$ O(3) SYMMETRY $\quad O(3)/O(2) = S^2$

$d=2$

$$E = \frac{1}{2}\int d^2x \; \partial_i \vec{\phi}\cdot\partial_i\vec{\phi} \qquad \text{with } \vec{\phi}\cdot\vec{\phi}=1$$



FINITE ENERGY

$\downarrow$

$\vec{\phi} = \text{CONST}$

AT INFINITY

SO MAPS ARE CLASSIFYED BY $\pi_2(S^2)$

$$\mathbb{R}^2 \cup \{\infty\} \longrightarrow S^2 \qquad \infty \longrightarrow y_0$$

$$N = \frac{1}{8\pi}\int \vec{\phi}\cdot\left(\partial_1\vec{\phi}\wedge\partial_2\vec{\phi}\right) d^2x \qquad \deg\psi$$

DEGREE OF THE MAP

BOGOMOLN'Y TRICK

CONTRACTION i, a

$$\left(\partial_i\vec{\phi} \pm \epsilon_{ij}\,\vec{\phi}\wedge\partial_j\vec{\phi}\right)^2 \geq 0$$

$$\partial_i\vec{\phi}\cdot\partial_i\vec{\phi} \pm 2\,\epsilon_{ij}\,\partial_i\vec{\phi}\cdot\left(\vec{\phi}\wedge\partial_j\vec{\phi}\right)$$
$$+\,\epsilon_{ij}\left(\vec{\phi}\wedge\partial_j\vec{\phi}\right)\cdot\epsilon_{ik}\left(\vec{\phi}\wedge\partial_k\vec{\phi}\right) \geq 0$$

$$\epsilon_{ij}\,\epsilon_{ik} = \delta_{jk}$$

$$2\,\partial_i\vec{\phi}\cdot\partial_i\vec{\phi} \mp \underbrace{2\,\epsilon_{ij}\,\vec{\phi}\cdot\left(\partial_i\vec{\phi}\wedge\partial_j\vec{\phi}\right)}_{\text{TOPOLOGICAL TERM}} \geq 0$$

$$\int d^2x\,\frac{1}{2}\,\partial_i\vec{\phi}\cdot\partial_i\vec{\phi} \geq \underbrace{\left|\frac{1}{2}\int d^2x\,\epsilon_{ij}\,\vec{\phi}\cdot\left(\partial_i\vec{\phi}\wedge\partial_j\vec{\phi}\right)\right|}_{=\,4\pi|n|}$$

$$n \longrightarrow \deg\phi$$

BOGO. BOUND $\quad 4\pi|n|$

BOGO. EQUATION $\quad \partial_i\vec{\phi} \pm \epsilon_{ij}\,\vec{\phi}\wedge\partial_j\vec{\phi} = 0$

FIRST ORDER

$$\partial_i\partial_i\vec{\phi} \pm \epsilon_{ij}\,\partial_i\left(\vec{\phi}\wedge\partial_j\vec{\phi}\right) = 0$$

$$\epsilon_{ij}\,\partial_i\partial_j\vec{\phi} = 0$$

$$\partial_i\partial_i\vec{\phi} \pm \epsilon_{ij}\left(\partial_i\vec{\phi}\wedge\partial_j\vec{\phi}\right) = 0$$

USE AGAIN BOGO $\quad \partial_i\vec{\phi} = \mp\,\epsilon_{ik}\,\vec{\phi}\wedge\partial_k\vec{\phi}$

$$\partial_i\partial_i\vec{\phi} \underbrace{-\,\epsilon_{ij}\,\epsilon_{ik}\left(\vec{\phi}\wedge\partial_k\vec{\phi}\right)\wedge\partial_j\vec{\phi}}_{+\vec{\phi}\left(\partial_i\vec{\phi}\cdot\partial_i\vec{\phi}\right)} = 0$$

BOGO. $\Rightarrow$ E-L EQUATION

E_L

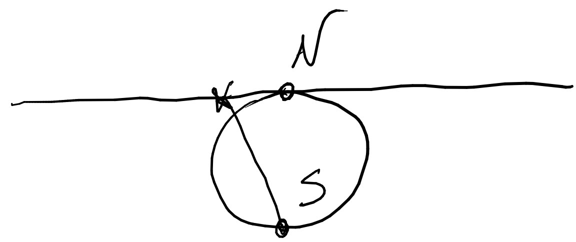
$$\partial_i \partial_i \vec{\phi} = \alpha \vec{\phi}$$

$$\vec{\phi} \cdot \partial_i \partial_i \vec{\phi} = \alpha = - \partial_i \vec{\phi}\, \partial_i \vec{\phi}$$

$$\partial_i \partial_i \vec{\phi} + (\partial_i \vec{\phi} \cdot \partial_i \vec{\phi})\, \vec{\phi} = 0$$

COMPLEX FORMULATION

$$R = \frac{\phi_1 + i \phi_2}{1 + \phi_3}$$



STEREOGRAPHIC PROJECTION

$$S^2 \equiv \mathbb{CP}^1$$

THEN WE ALSO USE COMPLEX VARIABLES FOR SPACE

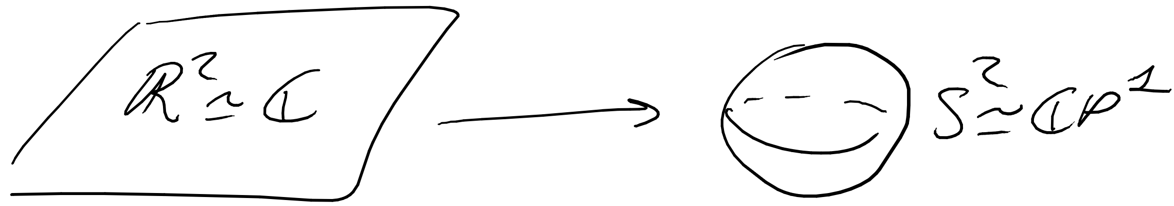
$z = x_1 + i x_2$, $\bar{z} = x_1 - i x_2$

$$\partial_z = \frac{1}{2}(\partial_1 - i \partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i \partial_2)$$

$$\partial_z z = 1, \quad \partial_z \bar{z} = 0, \quad \partial_{\bar{z}} z = 0, \quad \partial_{\bar{z}} \bar{z} = 1$$

$\partial_{\bar{z}} f = 0 \Rightarrow$ holomorphic $\quad \partial_z f = 0$ anti-holo

NOW WE HAVE COMPLEX MAX



$$\mathcal{L} = 2\,\frac{\partial_\mu R\, \partial^\mu \bar{R}}{(1+|R|^2)^2}$$

Metric $\mathbb{CP}^1$

$$E = 4\int d^2x\ \frac{\partial_z R\, \partial_{\bar z} \bar{R} + \partial_{\bar z} R\, \partial_z \bar{R}}{(1+|R|^2)^2}$$ ENERGY

$$N = \frac{1}{\pi}\int d^2x\ \frac{\partial_z R\, \partial_{\bar z} \bar{R} - \partial_{\bar z} R\, \partial_z \bar{R}}{(1+|R|^2)^2}$$ TOPOLOGICAL NUMBER

$\partial_{\bar z} R = 0$ HOLOMORPHIC (BOGO. EQUATION WITH + SIGN)

$E = 4\pi N$

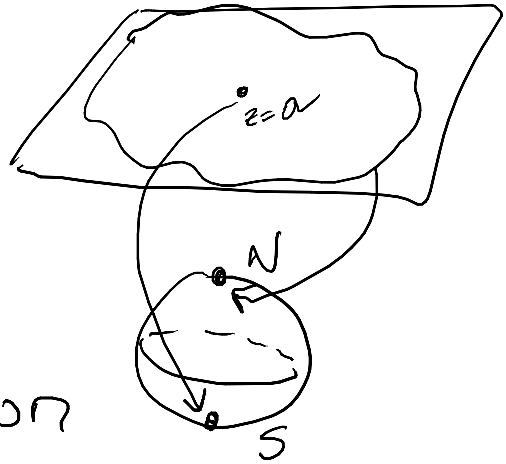
$\partial_z R = 0$ ANTI-HOL (BOGO. EQUATION WITH −)

$E = -4\pi N$

NOW BOGO. EQUATION ARE LINEAR EQUATIONS!

$$\partial_{\bar z} R = 0 \Rightarrow R = \frac{P(z)}{q(z)}$$ RATIONAL MAP.

$$R(z) = \frac{\lambda e^{i\alpha}}{z-a}$$



λ, α : SCALE & PHASE

a : POSITION

$\underline{\underline{4}}$ DEGREES OF FREEDOM FOR ONE TOPOLOGICAL CHARGE

$$R(z) = \frac{\beta_{q-1} z^{q-1} + \cdots + \beta_1 z + \beta_0}{z^q + \alpha_{q-1} z^{q-1} + \cdots + \alpha_1 z + \alpha_0}$$

$$R(z \to \infty) = N$$

DEGREES OF FREEDOM = 4 REAL

$$N = \max\{\deg p, \deg q\}$$

SKYRMION IN 3+1

$$L = \int d^3x \left(\frac{1}{2} \mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \underbrace{+\frac{1}{16} \mathrm{tr}\left([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2 \right)}_{\text{Skyrme term } E_4 \text{ TO HAVE STABILITY}} \right)$$

$U \in SU(2) \simeq S^3$

$\pi_3(S^3) = \mathbb{Z}$ TOPOLOGY FINITE ENERGY CONFIGURATIONS

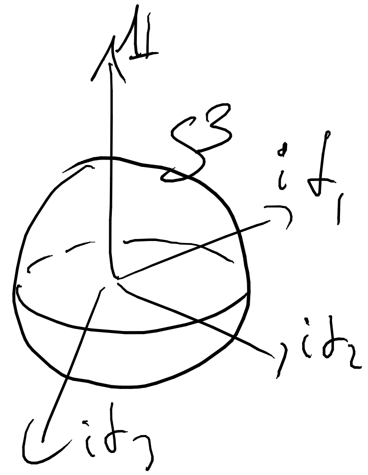
$$U = \phi_4 \mathbb{1} + i \phi_i \sigma_i = \begin{pmatrix} \phi_4 + i\phi_3 & i\phi_1 - \phi_2 \\ i\phi_1 + \phi_2 & \phi_4 - i\phi_3 \end{pmatrix}$$

$U^\dagger U = 1$

$\det U = 1$ $\Longleftrightarrow$ S^3, $\phi^a \phi^a = 1$

$SU(2)$

$$\frac{1}{2} \mathrm{tr}\left(\partial U \partial U^\dagger\right) = \partial_\mu \phi^a \partial^\mu \phi^a$$



$$N = \frac{1}{24\pi^2} \int d^3x \, \epsilon_{ijk} \, \mathrm{tr}\left(\partial_i U U^\dagger \partial_j U U^\dagger \partial_k U U^\dagger\right)$$

$$SO(4) \xrightarrow{\langle \phi \rangle} SO(3)$$

$$SU(2)_L \times SU(2)_R \longrightarrow SU(2)_D$$

CHIRAL SYMM. BREAKING

SKYRMION ANSATZ

$$U(r) = e^{i f(r) \sigma^a \hat{x}^a}$$

$$f(0) = 0$$

$$f(\infty) = 2\pi$$

THIS IS $[1] \in \pi_3(S^3)$